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Non-commutative tachyon action and D-brane geometry

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Abstract: We analyse open string correlators in non-constant background fields, including the metric g, the antisymmetric B-field, and the gauge field A. Working with a derivative expansion for the background fields, but exact in their constant parts, we obtain a tachyonic on-shell condition for the inserted functions and extract the kinetic term for the tachyon action. The 3-point correlator yields a non-commutative tachyon potential. We also find a remarkable feature of the differential structure on the D-brane: Although the boundary metric G plays an essential role in the action, the natural connection on the D-brane is the same as in closed string theory, i.e. it is compatible with the bulk metric and has torsion $H = dB$. This means, in particular, that the parallel transport on the brane is independent of the gauge field A.

Keywords: D-branes, Tachyon Condensation, Non-Commutative Geometry.

Contents

1. Introduction

The geometry on a D-brane has recently attracted much attention, as it turned out to involve non-commutative structures that depend on the gauge invariant combination $\mathcal{F} = B + (2\pi\alpha')dA$ of the bulk B-field and the boundary A-field. A lot of work was done in examining the effect of a constant B-field in a topological decoupling limit [1]–[5].

A generalization to non-constant fields was given through the deformation quantization of Poisson manifolds. In that case, a non-commutative product can be constructed to all orders of derivatives out of the Poisson structure Θ and it represents the most general form being associative [6]. In open string theory this product appears in the decoupling limit when the B-field (or equivalently $\mathcal{F}$) is closed [7].

While the closure condition is necessary for associativity, it is not required by string theory and one may ask how far one can relax it in order to obtain a reasonable product. In [8] the non-commutative product was extracted from open string off-shell correlators with insertions on the boundary of the disk. It turned out that one has to abandon the decoupling limit in order to retain a consistent setup. The only physical condition on the non-commutative parameter Θ in first derivative order of the background fields is the on-shell condition for the open string gauge field A on the D-brane, i.e. the generalized Maxwell equation; see also [9]–[12] for other attempts of treating a general background field B.

Imposing this equation has the following consequences for the product in first derivative order [8]. Firstly, the non-commutative product of two functions equals the ordinary product under the integral.[1] Secondly, the product is associative up to a surface term. As an immediate consequence, the product of an arbitrary number of functions is invariant

[1] In [8] it was only shown that $\int f \circ g = \int g \circ f$, but it is easily checked that, in fact, $\int f \circ g = \int fg$.

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under cyclic permutations under the integral up to a possible change in the bracket structure). The integration measure plays an important role in this respect and is given by a Born-Infeld measure, $\sqrt{\det(g-\mathcal{F})}$. Associativity cannot be maintained and must be replaced by an A_∞-structure [9, 14]. Both properties are necessary to construct a reasonable action and a variation principle in terms of the non-commutative product, the second one, to adjust the position of the variation of the field and the first one, to remove all derivatives from the variation.

So far only the non-commutative product arising in this generalized setting was considered. It can be extracted purely from off-shell correlation functions. However, since string vacua correspond to 2-d conformal field theories, the correlators must finally take shape of the usual simple on-shell form [15]. There is some interesting information which one can gain from passing to on-shell correlators and it is the intention of this article to work out this information.

For this purpose we will use several results from [8] and, therefore, inherit the general setting of the model considered there. The open strings move in a background including a general metric g and a nontrivial B-field in the bulk and a gauge field A on the boundary of the world sheet. The world sheet is taken to be the upper half complex plane. All information is extracted from correlation functions using a derivative expansion of the background fields, where the expansion is restricted to first derivative order, but exact to all orders in the constant part of $\mathcal{F}$.

What can we learn from the on-shell correlators?

i. As the insertions at the boundary of the disk are taken to be ordinary functions of the target space coordinates X^μ we expect that the on-shell condition is tachyonic. Furthermore, the equation of motion is linear in the tachyonic field since the insertions represent asymptotic states. We will deduce this linear equation from the requirement that the on-shell correlators must have the CFT form and thus obtain the kinetic term for the effective action of the tachyon.

ii. The explicit form of the on-shell three-point function then gives us the cubic interaction of the open string tachyon potential. Higher n-point correlators are difficult to manage. Nevertheless, in view of the cyclicity of the product under the integral, we are able to discuss some implications for the structure of higher order interactions.

iii. Working in first derivative order of the background fields is already sufficient to extract information about the differential structure from the tachyon equation of motion. As an interesting and somewhat surprising result we anticipate that, using the generalized Maxwell equation, the covariant derivative on the D-brane turns out to be the same as that off the brane, i.e. it is the connection compatible with the bulk metric and with torsion $H = dB$.

The organisation of the paper is as follows. We start in section 2 with the introduction of some notation and review the properties of the non-commutative product. In section 3 we calculate the full two- and three-point off-shell correlators using results from [8]. Thereafter, in section 4, we show that conformal invariance requires a tachyonic on-shell condition for the insertions of the correlators and the use of the Maxwell equation for the background fields. Eventually, we investigate the potential and the differential structure of the tachyonic action in section 5 and close with a general discussion of our results in section 6.

2. The non-commutative product

On the D-brane we have in addition to the so called bulk metric $g_{\mu\nu}$, which enters in the sigma model action, the boundary metric $G^{\mu\nu}$. Two metrics arise because of the fact that on the brane one has to consider the combination $M_{\mu\nu} = g_{\mu\nu} + \mathcal{F}_{\mu\nu}$ rather than the separate quantities g and $\mathcal{F}$. Consequently, one can split the inverse of M into the symmetric and antisymmetric part, i.e. $M^{\mu\nu} := M^{-1\mu\nu} = G^{\mu\nu} + \Theta^{\mu\nu}$, and obtains the second metric G and the antisymmetric part Θ, which turns out to be the non-commutativity parameter.[2]

The product found in [8] is given to all orders in Θ and to first derivative order in the background fields. It reads

$$f(x)\ \circ\ g(x) = f*g - \frac{1}{12}\Theta^{\mu\rho}\partial_\rho\Theta^{\nu\sigma}\left(\partial_\mu\partial_\nu f\ *\partial_\sigma g + \partial_\sigma f\ *\partial_\mu\partial_\nu g\right) + \mathcal{O}((\partial\Theta)^2, \partial^2\Theta)\,, \quad (2.1)$$

where '$*$' denotes the Moyal contribution to the product. The on-shell condition for the gauge field A on the D-brane is

$$G^{\rho\sigma}D_\rho\mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma}H_{\rho\sigma}{}^{\lambda}\mathcal{F}_{\lambda\mu} = 0\,, \quad (2.2)$$

where D_ρ is the Christoffel connection of g, or equivalently,

$$\partial_\mu\left(\sqrt{g-\mathcal{F}}\ \Theta^{\mu\nu}\right) = 0\,. \quad (2.3)$$

Imposing this equation of motion one finds that the product of two functions equals the ordinary product under the integral,

$$\int \mathrm{d}^D x\sqrt{\det(g-\mathcal{F})}\ f\circ g = \int \mathrm{d}^D x\sqrt{\det(g-\mathcal{F})}\ f\cdot g\,, \quad (2.4)$$

and that it is associative up to a surface term,

$$\int \mathrm{d}^D x\sqrt{\det(g-\mathcal{F})}\ (f\circ g)\circ h = \int \mathrm{d}^D x\sqrt{\det(g-\mathcal{F})}\ f\circ(g\circ h)\,. \quad (2.5)$$

The trace property

$$\begin{aligned}\int \mathrm{d}^D x\sqrt{\det(g-\mathcal{F})}&\Big((\ldots(f_1\circ\ldots))\circ f_{n-1}\Big)\circ f_n = \\ &\int \mathrm{d}^D x\sqrt{\det(g-\mathcal{F})}\Big(f_n\circ(\ldots(f_1\circ\ldots))\Big)\circ f_{n-1}\,. \end{aligned}\quad (2.6)$$

follows immediately. Although the integration measure plays an important role in order to derive these properties we will subsequently use the abbreviation $\int_x = \int \mathrm{d}^D x\sqrt{\det(g-\mathcal{F})}$ for the integral. These results will extensively be used in sections 4 and 5.

[2] We use the convention $M^{-1\mu\nu}M_{\nu\rho} = \delta^\mu{}_\rho$.

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3. Off-shell correlators

We are now going to use several results of the appendix of [8] to calculate the full off-shell two- and three-point correlators of ordinary functions of target space coordinates X^μ. The insertions are ordered at the boundary of the upper half plane, so that $\tau_1 < \tau_2 < \tau_3$.

The two-point correlator was already given in [8] and we repeat the expression in a more compact form. To this end we use the relation

$$2G^{\nu\rho}G^{\sigma\lambda}\bar{\Gamma}_{\rho\lambda}{}^{\mu} = G^{\mu\rho}\partial_\rho G^{\nu\sigma} - G^{\nu\rho}\partial_\rho G^{\sigma\mu} - G^{\sigma\rho}\partial_\rho G^{\nu\mu}\,, \tag{3.1}$$

to introduce the Christoffel connection compatible with G. With the abbreviation ($\tau_{ij} = \tau_i - \tau_j$)

$$f\ \Delta\ g := \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{1}{2\pi}\Big)^n \ln^n \tau_{21}^{-2}\ G^{\mu^n\nu^n}\ \bar{D}_{\mu^n} f\ \circ\ \bar{D}_{\nu^n} g \tag{3.2}$$

where the upper subindices of the indices mean a product of derivatives,

$$\bar{D}_{\mu^n} := \bar{D}_{(\mu_1}\ldots\bar{D}_{\mu_n)} = \partial_{\mu^n} f - \frac{n(n-1)}{2}\,\bar{\Gamma}_{(\mu_1\mu_2}{}^{\rho}\partial_{\mu_3}\ldots\partial_{\mu_n)}\partial_\rho f + \mathcal{O}(\partial^2)$$

and a product of metrics $G^{\mu^n\nu^n} = G^{\mu_1\nu_1}\ldots G^{\mu_n\nu_n}$. The symmetrization of the derivatives $\bar{D}_\mu$ in $\bar{D}_{\mu^n}$ is automatic in first order, because the partial derivatives contracted with G symmetrize the $\bar{\Gamma}$-term on the other side, i.e. $G^{\mu\nu}\bar{D}_\mu f\bar{D}_\nu g = G^{\mu\nu}\partial_\mu f\bar{D}_\nu g + G^{\mu\nu}\bar{D}_\mu f\partial_\nu g + \mathcal{O}(\partial^2)$. We can then write the full two-point correlator as

$$\begin{aligned}\langle :f[X(\tau_1)]::g[X(\tau_2)]:\rangle \ = \quad & \\ \int_x f\ \Delta\ g + \frac{i}{4\pi}\ln\tau_{21}^{-1}\int_x \Theta^{\mu\rho}\partial_\rho G^{\nu\sigma}&(\partial_\nu\partial_\sigma f\ \Delta\ \partial_\mu g - \partial_\mu f\ \Delta\ \partial_\nu\partial_\sigma g) + \\ +\frac{i}{2\pi}\ln\tau_{21}^{-1}\int_x G^{\nu\sigma}\partial_\sigma\Theta^{\rho\mu}&(\partial_\nu\partial_\rho f\ \Delta\ \partial_\mu g - \partial_\mu f\ \Delta\ \partial_\nu\partial_\rho g) + \mathcal{O}(\partial^2)\,.\end{aligned} \tag{3.3}$$

The three-point correlator is much more complicated. It is a rather tedious but straightforward work to collect all the relevant terms from the appendix in [8]. In order to realize the structure more clearly we first consider the two cases, $\Theta = 0$ and $\Theta \to \infty$ (or, equivalently, $G \to 0$). For $\Theta = 0$, the covariant derivative $\bar{D}$ again appears, now in the combination

$$\begin{aligned}\bar{D}_{\mu^n}\bar{D}_{\nu^m} f = {} & \partial_{\mu^n}\partial_{\nu^m} f - \\ & - \frac{n(n-1)}{2}\,\bar{\Gamma}_{(\mu_1\mu_2}{}^{\rho}\partial_{\mu_3}\ldots\partial_{\mu_n)}\partial_{\nu^m}\partial_\rho f - \\ & - \frac{m(m-1)}{2}\,\bar{\Gamma}_{(\nu_1\nu_2}{}^{\rho}\partial_{\nu_3}\ldots\partial_{\nu_m)}\partial_{\mu^n}\partial_\rho f - \\ & - mn\,\bar{\Gamma}_{(\mu_1|(\nu_1}{}^{\rho}\partial_{\nu_2}\ldots\partial_{\nu_m)}\partial_{|\mu_2}\ldots\partial_{\mu_n)}\partial_\rho f + \mathcal{O}(\partial^2)\,.\end{aligned} \tag{3.4}$$

The correlator is given as

$$
\begin{aligned}
\langle :f[X(\tau_1)]::g[X(\tau_2)]::h[X(\tau_3)]:\rangle|_{\Theta=0} &= \int_x \Delta(f,g,h) + \mathcal{O}(\partial^2) = \\
&\sum_{I,J,K}\left(\frac{1}{\pi}\right)^{I+J+K} \frac{\ln^I \tau_{21}^{-1} \ln^J \tau_{31}^{-1} \ln^K \tau_{32}^{-1}}{I!J!K!} \times \\
&\times \int_x G^{\mu^I\nu^I} G^{\mu^J\nu^J} G^{\mu^K\nu^K} \bar{D}_{\mu^I\mu^J} f\ \bar{D}_{\nu^I\mu^K} g\ \bar{D}_{\nu^J\nu^K} h + \mathcal{O}(\partial^2)\,. \quad (3.5)
\end{aligned}
$$

The triangle functional $\Delta(f,g,h)$ in the first line is a generalization of (3.2) and was introduced for later reference. In case of $G \to 0$ the correlator can be expressed solely through the non-commutative product (2.1) (cf. also [9])

$$
\begin{aligned}
\langle :f[X_1]::g[X_2]::h[X_3]:\rangle|_{G=0} = &\int_x \frac{1}{2}\Big\{(f\circ g)\circ h + f\circ(g\circ h) + \\
&+ L(m)\,(f_1\circ(f_2\circ f_3) - (f_1\circ f_2)\circ f_3)\Big\} + \\
&+ \mathcal{O}(\partial^2)\,.
\end{aligned}
$$

$L(m) = \frac{6}{\pi^2}(\mathrm{Li}_2(m) - \mathrm{Li}_2(1-m))$ is an antisymmetric combination of dilogarithms $\mathrm{Li}_2(m)$ with the limits $L(0) = -1$ and $L(1) = 1$. The modulus $m = \tau_{21}/\tau_{31}$ can take the values $0 \le m \le 1$.

Since we take into account only terms to first derivative order these two results can easily be completed to the general case. In equations (3.5) and (3.6) we have included the $G\partial G$- and the $\Theta\partial\Theta$-terms, respectively. In the full correlator these two results combine in a natural way and add up with the remaining $G\partial\Theta$ and $\Theta\partial G$ parts, so that we find

$$
\langle :f[X(\tau_1)]::g[X(\tau_2)]::h[X(\tau_3)]:\rangle = F[G\partial G, \Theta\partial\Theta] + F[G\partial\Theta, \Theta\partial G] + \mathcal{O}(\partial^2) \quad (3.6)
$$

where

$$
\begin{aligned}
F[G\partial G, \Theta\partial\Theta] = &\sum_{I,J,K} \frac{\ln^I \tau_{21}^{-1} \ln^J \tau_{31}^{-1} \ln^K \tau_{32}^{-1}}{I!J!K!} \times \quad (3.7) \\
&\times \frac{1}{\pi^{I+J+K}} \int_x G^{\mu^I\nu^I} G^{\mu^J\nu^J} G^{\mu^K\nu^K} \frac{1}{2}\Big[\bar{D}_{\mu^I\mu^J} f \circ (\bar{D}_{\nu^I\mu^K} g \circ \bar{D}_{\nu^J\nu^K} h) + \\
&+ (\bar{D}_{\mu^I\mu^J} f \circ \bar{D}_{\nu^I\mu^K} g) \circ \bar{D}_{\nu^J\nu^K} h + \\
&+ L(m)\Big(\bar{D}_{\mu^I\mu^J} f \circ (\bar{D}_{\nu^I\mu^K} g \circ \bar{D}_{\nu^J\nu^K} h) - \\
&- (\bar{D}_{\mu^I\mu^J} f \circ \bar{D}_{\nu^I\mu^K} g) \circ \bar{D}_{\nu^J\nu^K} h\Big)\Big]
\end{aligned}
$$

and, with $\partial_\mu f = f_\mu$,

$$
\begin{aligned}
&F[G\partial\Theta, \Theta\partial G] = \\
&\quad + \frac{i}{2\pi}\ln\tau_{21}^{-1}\int_x \Theta^{\rho\sigma}\partial_\sigma G^{\mu\nu}\left[-\Delta(f_\mu, g_\nu, h_\rho) - \frac{1}{2}\Delta(f_\rho, g_{\mu\nu}, h) + \frac{1}{2}\Delta(f_{\mu\nu}, g_\rho, h)\right] + \\
&\quad + \frac{i}{2\pi}\ln\tau_{31}^{-1}\int_x \Theta^{\rho\sigma}\partial_\sigma G^{\mu\nu}\Big[-\Delta(f_\rho, g_\mu, h_\nu) + \Delta(f_\mu, g_\nu, h_\rho) - \\
&\qquad\qquad\qquad - \frac{1}{2}\Delta(f_\rho, g, h_{\mu\nu}) + \frac{1}{2}\Delta(f_{\mu\nu}, g, h_\rho)\Big] + \\
&\quad + \frac{i}{2\pi}\ln\tau_{32}^{-1}\int_x \Theta^{\rho\sigma}\partial_\sigma G^{\mu\nu}\left[+\Delta(f_\rho, g_\mu, h_\nu) - \frac{1}{2}\Delta(f, g_\rho, h_{\mu\nu}) + \frac{1}{2}\Delta(f, g_{\mu\nu}, h_\rho)\right] + \\
&\quad + \frac{i}{2\pi}\ln\tau_{21}^{-1}\int_x G^{\rho\sigma}\partial_\sigma\Theta^{\mu\nu}[+\Delta(f_\mu, g_{\nu\rho}, h) - \Delta(f_{\nu\rho}, g_\mu, h)] + \\
&\quad + \frac{i}{2\pi}\ln\tau_{31}^{-1}\int_x G^{\rho\sigma}\partial_\sigma\Theta^{\mu\nu}\Big[+\Delta(f_\mu, g_\nu, h_\rho) + \Delta(f_\nu, g_\rho, h_\mu) - \Delta(f_\rho, g_\mu, h_\nu) + \\
&\qquad\qquad\qquad + \Delta(f_\mu, g, h_{\nu\rho}) - \Delta(f_{\nu\rho}, g, h_\mu)\Big] + \\
&\quad + \frac{i}{2\pi}\ln\tau_{32}^{-1}\int_x G^{\rho\sigma}\partial_\sigma\Theta^{\mu\nu}[+\Delta(f, g_\mu, h_{\nu\rho}) - \Delta(f, g_{\nu\rho}, h_\mu)]\,.
\end{aligned}
\tag{3.8}
$$

The symbol $\Delta(f, g, h)$ is defined in equation (3.5).

4. On-shell correlators

In the previous section the insertions on the disk as well as the background fields are completely general, they do not satisfy any on-shell conditions, which are determined by the conformal invariance of the theory. The equations of motion for the background fields are given by the β-functions of the world sheet theory whereas the equations of motion for the insertions are determined by the conformal transformation properties of the correlation functions.

The correlators of the CFT on the disk must be invariant under the global conformal group $\mathrm{SL}(2,\mathbb{R})$. In particular, the 2-point correlator with insertions at the boundary is

$$\langle f_1[X(\tau_1)] f_2[X(\tau_2)]\rangle = \frac{C_{12}}{(\tau_{21})^{2h}}\,, \tag{4.1}$$

where h is the conformal weight of both f_1 and f_2. The correlator for operators with different weights vanishes. The 3-point correlator is

$$\langle f_1[X(\tau_1)] f_2[X(\tau_2)] f_3[X(\tau_3)]\rangle = \frac{C_{123}}{\tau_{21}^{h_1+h_2-h_3}\tau_{31}^{h_3+h_1-h_2}\tau_{32}^{h_2+h_3-h_1}} \tag{4.2}$$

The constants C_{12} and C_{123} are functionals of $f_i(x)$, independent of the positions τ_i and invariant under cyclic permutation of indices. For physical fields which should carry $h = 1$ we have

$$\langle f_1[X_1]\ f_2[X_2]\rangle = \frac{C_{12}}{(\tau_1 - \tau_2)^2}\,, \tag{4.3}$$

$$\langle f_1[X_1]\ f_2[X_2]\ f_3[X_3]\rangle = \frac{C_{123}}{\tau_{21}\tau_{31}\tau_{32}}\,. \tag{4.4}$$

On the other hand, the off-shell correlators in the open string background, (3.3) and (3.6), can be written in the following way,

$$\langle f_1[X_1]\ f_2[X_2]\rangle = \sum_{I=0}^{\infty} \frac{\ln^I \tau_{21}^{-2}}{I!} F_I[f_i](\tau_i) \tag{4.5}$$

$$\langle f_1[X_1]\ f_2[X_2]\ f_3[X_3]\rangle = \sum_{I,J,K=0}^{\infty} \frac{\ln^I \tau_{21}^{-1} \ln^J \tau_{31}^{-1} \ln^K \tau_{32}^{-1}}{I!J!K!} F_{IJK}[f_i](\tau_i)$$

where $F_I[f_i](\tau_i)$ and $F_{IJK}[f_i](\tau_i)$ are functionals of f_i and functions of τ_i. The τ_i-dependence arises from the dilogarithm in (3.7) as well as from the sign function $\epsilon(\tau_{ij})$ which accompanies every Θ (cf. [8]). In fact, we do not see the sign function because of our choice of ordering, $\tau_1 < \tau_2 < \tau_3$.

Therefore, if

$$F_I[f_i](\tau_i) = F_{I-1}[f_i](\tau_i) \tag{4.6}$$

$$F_{IJK}[f_i](\tau_i) = F_{(I-1)JK}[f_i](\tau_i) = F_{I(J-1)K}[f_i](\tau_i) = F_{IJ(K-1)}[f_i](\tau_i) \tag{4.7}$$

is fulfilled, one can reduce all functionals in the sum to $F_0[f_i](\tau_i)$ and $F_{000}[f_i](\tau_i)$, respectively. Furthermore, in order to reproduce the behaviour (4.3) and (4.4), $F_0[f_i](\tau_i)$ and $F_{000}[f_i](\tau_i)$ must be constants and then determine $C_{12} = F_0[f_i]$ and $C_{123} = F_{000}[f_i]$. However, this does not work off-shell and has to be accomplished by certain on-shell conditions imposed on the insertions (and of course on the background fields). We proceed in two steps and first show the following theorem:

Theorem 1 (Relations) *(4.6) and (4.7) require that the insertions satisfy the tachyonic equation of motion*

$$\Box f_i - (-2\pi) f_i = \frac{1}{\sqrt{\det(g-\mathcal{F})}}\ \partial_\mu(\sqrt{\det(g-\mathcal{F})} G^{\mu\nu} \partial_\nu f_i) - (-2\pi) f_i = 0\,. \tag{4.8}$$

Proof. We start with the 2-point correlator. In order to get a scalar equation of the insertions one has to integrate by part. On the other hand, if we look at S-matrix calculations [15] the momentum conservation ($\delta^D(\Sigma k)$) comes from the integration over zero modes. Here we do not have a flat background and we cannot perform the integration in that way. Now the integration by part is the analog of the momentum conservation in position space. Furthermore, one can separate the functionals F_I into two distinct parts, $F_I = F_I[\Theta d\Theta] + F_I[GdG, Gd\Theta, \Theta dG]$. The former one comes from the first term on the right hand side of (3.3), but without the Christoffel symbols, the latter arises from the rest of (3.3).

We take $F_I[\Theta d\Theta]$ and integrate by part in the following way

$$\begin{aligned} F_I[\Theta d\Theta] &= \left(\frac{1}{2\pi}\right)^I \int_x G^{\mu^I\nu^I} \partial_{\mu^I} f_1 \ \circ\ \partial_{\nu^I} f_2 \\ &= -\frac{1}{2}\left(\frac{1}{2\pi}\right)^I \int_x G^{\mu^{I-1}\nu^{I-1}} \partial_{\mu^{I-1}} \Box f_1 \ \circ\ \partial_{\nu^{I-1}} f_2 - \\ &\quad -\frac{1}{2}\left(\frac{1}{2\pi}\right)^I \int_x G^{\mu^{I-1}\nu^{I-1}} \partial_{\mu^{I-1}} f_1 \ \circ\ \partial_{\nu^{I-1}} \Box f_2 + \\ &\quad + F_I'[GdG, Gd\Theta, \Theta dG]\,. \end{aligned} \tag{4.9}$$

The last expression $F'_I[GdG, Gd\Theta, \Theta dG]$ combines with $F_I[GdG, Gd\Theta, \Theta dG]$. Analogously, using integration by part in $F''_I = F'_I[GdG, Gd\Theta, \Theta dG] + F_I[GdG, Gd\Theta, \Theta dG]$, the differential operator as it appears in (4.9) arises now in zero derivative order of the background fields, i.e. $\Box f_i = G^{\mu_I \nu_I}\partial_{\mu_I}\partial_{\nu_I} f_i$, because F''_I contains only terms of first derivative order. Now $F_{I-1}[\Theta d\Theta]$ is equal to the first two lines of (4.9) if equation (4.8) holds. Proceeding along the same lines one can show that $F_{I-1}[GdG, Gd\Theta, \Theta dG] = F''_I$, again using (4.8). In fact, the 2-point function does not fix the tachyonic equation uniquely. One can add $A^{\mu^{2n+1}}\partial_{\mu^{2n+1}} f_i$, where $n \in \mathbb{Z}$ and $A \sim \mathcal{O}(\partial)$. By means of partial integration such terms would mutually cancel in the second and third line of (4.9). However, we will see that this ambiguity is fixed by the 3-point correlator.

The calculation for the 3-point correlator is similar. First we make the split $F_{IJK} = F_{IJK}[\Theta d\Theta] + F_{IJK}[GdG, Gd\Theta, \Theta dG]$. Let us again look at the S-matrix calculation. There one uses the relation $k_1 k_2 = \frac{1}{4}(k_1 - k_2 - k_3)(k_2 - k_1 - k_3) = \frac{1}{2}(k_3^2 - k_1^2 - k_2^2)$. With the analogous transformation in terms of partial integrations we obtain from (3.6) and (3.7)

$$
\begin{aligned}
F_{IJK}[\Theta d\Theta] = & \\
= \frac{1}{\pi^{I+J+K}} \int_x G^{\mu^I \nu^I} G^{\mu^J \nu^J} G^{\mu^K \nu^K} \frac{1}{2} \Big[& \partial_{\mu^I \mu^J} f_1 \circ (\partial_{\nu^I \mu^K} f_2 \circ \partial_{\nu^J \nu^K} f_3) + \\
& + (\text{other bracket}) + \\
& + L(m) \Big(\partial_{\mu^I \mu^J} f_1 \circ (\partial_{\nu^I \mu^K} f_2 \circ \partial_{\nu^J \nu^K} f_3) - \\
& \quad - (\text{other bracket}) \Big) \Big] = \\
= \frac{1}{\pi^{I+J+K}} \int_x G^{\mu^{I-1} \nu^{I-1}} G^{\mu^J \nu^J} G^{\mu^K \nu^K} \frac{1}{4} \Big[& \partial_{\mu^{I-1} \mu^J} f_1 \circ (\partial_{\nu^{I-1} \mu^K} f_2 \circ \partial_{\nu^J \nu^K} \Box f_3) - \\
& - \partial_{\mu^{I-1} \mu^J} \Box f_1 \circ (\partial_{\nu^{I-1} \mu^K} f_2 \circ \partial_{\nu^J \nu^K} f_3) - \\
& - \partial_{\mu^{I-1} \mu^J} f_1 \circ (\partial_{\nu^{I-1} \mu^K} \Box f_2 \circ \partial_{\nu^J \nu^K} f_3) + \\
& + (\text{other bracket}) + \\
& + L(m) \Big(\partial_{\mu^{I-1} \mu^J} f_1 \circ (\partial_{\nu^{I-1} \mu^K} f_2 \circ \partial_{\nu^J \nu^K} \Box f_3) - \\
& \quad - \partial_{\mu^{I-1} \mu^J} \Box f_1 \circ (\partial_{\nu^{I-1} \mu^K} f_2 \circ \partial_{\nu^J \nu^K} f_3) - \\
& \quad - \partial_{\mu^{I-1} \mu^J} f_1 \circ (\partial_{\nu^{I-1} \mu^K} \Box f_2 \circ \partial_{\nu^J \nu^K} f_3) - \\
& \quad - (\text{other bracket}) \Big) \Big] + \\
& \quad + F'_{IJK}[GdG, Gd\Theta, \Theta dG] \,.
\end{aligned} \tag{4.10}
$$

Since the dilogarithmic term is of first derivative order in background fields there are no contributions thereof in $F'_{IJK}[GdG, Gd\Theta, \Theta dG]$. The same procedure as above shows that in view of (4.8) the first part of (4.10) equals $F_{(I-1)JK}[\Theta d\Theta]$ and $F'_{IJK}[GdG, Gd\Theta, \Theta dG] + F_{IJK}[GdG, Gd\Theta, \Theta dG] = F_{(I-1)JK}[GdG, Gd\Theta, \Theta dG]$. But now terms like $A^{\mu^{2n+1}}\partial_{\mu^{2n+1}} f_i$ would not cancel in (4.10), so that the tachyonic equation of motion (4.8) is unique. Relations $F_{IJK} = F_{I(J-1)K}$ and $F_{IJK} = F_{IJ(K-1)}$ can be shown analogously. $\Box$

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With this result we can write the correlators as

$$\begin{aligned}
\langle f_1[X_1]\ f_2[X_2]\rangle &= \frac{F_0[f_i](\tau_i)}{{\tau_{21}}^2} && (4.11)\\
\langle f_1[X_1]\ f_2[X_2]\ f_3[X_3]\rangle &= \frac{F_{000}[f_i](\tau_i)}{\tau_{21}\tau_{31}\tau_{32}}
\end{aligned}$$

with

$$F_0[f_i](\tau_i) = \int_x (f_1 \circ f_2)\,, \tag{4.12}$$

$$\begin{aligned}
F_{000}[f_i](\tau_i) = \int_x \frac{1}{2}\Big\{ & f_1 \circ (f_2 \circ f_3) + (f_1 \circ f_2) \circ f_3 + \qquad (4.13)\\
& + L(m)\Big(f_1 \circ (f_2 \circ f_3) - (f_1 \circ f_2) \circ f_3\Big)\Big\}\,.
\end{aligned}$$

Indeed, (4.12) and (4.13) are not yet position independent and invariant under cyclic exchange of the functions f_i. Putting also the background fields on shell, i.e. using the Maxwell equation (2.3), we can take advantage of the relations (2.4) and (2.5). So, we reach the final result

$$\begin{aligned}
\langle f_1[X_1]\ f_2[X_2]\rangle &= \frac{1}{{\tau_{21}}^2}\int_x f_1 \circ f_2 = \frac{1}{{\tau_{21}}^2}\int_x f_1 \cdot f_2\,, && (4.14)\\
\langle f_1[X_1]\ f_2[X_2]\ f_3[X_3]\rangle &= \frac{1}{\tau_{21}\tau_{31}\tau_{32}}\int_x f_1 \circ f_2 \circ f_3\,.
\end{aligned}$$

We close this section with a remark on the ghost fields, which we have totally excluded from our discussion so far. On the disc we have three conformal killing vectors (forming the Möbius group $\mathrm{SL}(2,\mathbb{R})$) and therefore three of the vertices in a correlator can be fixed in position and must be accompanied by a ghost field $c(\tau_i)$, the others being integrated over the world sheet. The 2-point correlator has too few insertions in order to give a non-vanishing result in the ghost sector, i.e. $\langle c(\tau_1)c(\tau_2)\rangle_{gh} = 0$. The 3-point ghost amplitude, $\langle c(\tau_1)c(\tau_2)c(\tau_3)\rangle_{gh} = c_{gh}\ \tau_{21}\tau_{31}\tau_{32}$, exactly cancels the position dependence of the correlator (4.4). Moreover, the Möbius group preserves the cyclic order of the insertions and so we must sum over inequivalent orderings in the 3-point amplitude, so that we obtain[3]

$$\langle cf_1[X_1]\ cf_2[X_2]\ cf_3[X_3]\rangle + (f_2 \leftrightarrow f_3) = c\int_x f_1 \circ (f_2 \circ f_3 + f_3 \circ f_2)\,. \tag{4.15}$$

5. Tachyonic action

The results (4.8) and (4.15) enable us to reconstruct the kinetic term and the cubic potential of the open string tachyon. The value for the coupling constant is recovered from

[3]Until now we have chosen the normalization of the correlators such that it reduces to an integral over the ordinary product of functions for $\mathcal{F} = 0$. We reintroduce a normalization constant $c = c_X c_{gh}$ where c_X and c_{gh} are the normalizations for the matter and the ghost contribution, respectively. c is fixed by unitarity [15].

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consistency with S-matrix calculations, as discussed e.g. in [15], by taking the limit $\Theta \to 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$.

$$S = -\frac{1}{2g_o^2}\int d^D x\sqrt{g-\mathcal{F}}\ \left\{G^{\mu\nu}\cdot\partial_\mu T\cdot\partial_\nu T - \frac{1}{\alpha'}\ T\cdot T - \sqrt{\frac{8}{9\alpha'}}\ T\cdot(T\circ T)\right\}. \tag{5.1}$$

It would be more natural if we write the action totally in terms of the non-commutative product (2.1), i.e.

$$S = -\frac{1}{2g_o^2}\int d^D x\sqrt{g-\mathcal{F}}\ \left\{G^{\mu\nu}\circ\partial_\mu T\circ\partial_\nu T - \frac{1}{\alpha'}\ T\circ T - \sqrt{\frac{8}{9\alpha'}}\ T\circ T\circ T\right\}. \tag{5.2}$$

The kinetic term of this action generates the equation of motion

$$\frac{1}{2}\frac{1}{\sqrt{g-\mathcal{F}}}\ \partial_\mu\Big(\sqrt{g-\mathcal{F}}\ (G^{\mu\nu}\circ\partial_\nu T + \partial_\nu T\circ G^{\mu\nu})\Big) + \frac{1}{\alpha'}T = 0\,, \tag{5.3}$$

which reduces to equation (4.8) because $\frac{1}{2}(G^{\mu\nu}\circ\partial_\nu T + \partial_\nu T\circ G^{\mu\nu}) = G^{\mu\nu}\partial_\nu T + \mathcal{O}(\partial^2 G)$. This means that the question whether one has to put the non-commutative product into the kinetic term or not cannot be decided at first derivative order.

If we impose the background field on-shell condition (2.3), the kinetic term of (4.8) reveals a remarkable feature of the geometry on the D-brane. Equation (2.3) implies also $\partial_\mu(\sqrt{g-\mathcal{F}}G^{\mu\nu}) = \sqrt{g-\mathcal{F}}M^{\rho\sigma}(-\Gamma_{\rho\sigma}{}^\nu - \frac{1}{2}H_{\rho\sigma}{}^\nu)$, and we are able to rewrite (4.8) as

$$M^{\mu\nu}\nabla_\mu\nabla_\nu T - (-2\pi)T = 0\,, \tag{5.4}$$

where we have introduced the connection ∇ that is compatible with the bulk metric and has torsion H

$$\nabla_\mu\xi_\nu = \partial_\mu\xi_\nu - \Gamma_{\mu\nu}{}^\rho\xi_\rho - \frac{1}{2}H_{\mu\nu}{}^\rho\xi_\rho\,. \tag{5.5}$$

This is exactly the connection that appears in closed string theory and it is independent of the gauge field A.

Finally, we make a remark on higher order interactions in the tachyonic potential. Since each term in the potential is a power of the field T, it is a very symmetric expression and one may ask if more brackets than the outermost can be omitted and, if so, how many. For "$T^{\circ n}$" with $n = 4, 5$ it is easy to show that all brackets can be left out, i.e. $\int_x T^{\circ 4} = \int_x T\circ T\circ T\circ T$ and $\int_x T^{\circ 5} = \int_x T\circ T\circ T\circ T\circ T$. What happens if we vary these expressions? To this end one has to select an arbitrary choice for the brackets in $T^{\circ 4}$ and $T^{\circ 5}$. Independently of that choice the variation gives the sum over all bracket arrangements for three and four functions,

$$\delta\int_x T^{\circ 4} = \int_x \delta T\cdot\Big(T\circ(T\circ T)) + ((T\circ T)\circ T)\Big) \tag{5.6}$$

$$\begin{aligned}\delta\int_x T^{\circ 5} = \int_x \delta T\cdot\Big(&T\circ(T\circ(T\circ T)) + T\circ((T\circ T)\circ T) + (T\circ T)\circ(T\circ T) + \\ &+ (T\circ(T\circ T))\circ T + ((T\circ T)\circ T)\circ T\Big)\,.\end{aligned} \tag{5.7}$$

For higher powers the behaviour is different. For instance, in case of $n = 6$ the following four expressions have different variations:

$$\begin{aligned} &\int_x (T \circ (T \circ T)) \circ ((T \circ T) \circ T), \\ &\int_x (T \circ (T \circ T)) \circ (T \circ (T \circ T)), \\ &\int_x ((T \circ T) \circ T) \circ ((T \circ T) \circ T), \\ &\int_x ((T \circ T) \circ (T \circ T)) \circ (T \circ T). \end{aligned} \tag{5.8}$$

Any other bracket arrangements can be converted to one of these possibilities by means of equations (2.4) and (2.5). The variations of (5.8) give a sum of 6, 3, 3, and 2 different terms, respectively.[4] Therefore, for higher powers than five we have distinct expressions, where either one could arise in the tachyon potential, possibly with different weights.

6. Discussion

In view of the quantity $M_{\mu\nu}$ and its inverse, the two metrics, g and G, seem to be on equal footing on the D-brane. For instance, the integration measure can be written as $\sqrt[4]{g}\sqrt[4]{G}$. However, in the effective field theory on the brane they play different roles. G appears in the kinetic term of the action and, therefore, one expects that it is the preferred metric on the brane. However, as we have seen in section 5, the natural connection on the D-brane is compatible with g and has torsion $H = dB$. This has an interesting consequence. The connection and the parallel transport is independent of the open string gauge field A and depends only on bulk quantities. So we have gained some insight into the differential structure on a D-brane and it would be even more interesting to see how this extends to second derivative order. The correlation functions of open string photon vertices instead of tachyon vertices would be another source of information. They should give rise to a gauge theory in a general non-commutative background.

We have seen that the form of the terms in the tachyonic potential is determined for powers lower than six. For higher powers of the tachyon field one has to calculate the corresponding correlators in order to decide with which relative weight the distinct subsets (e.g. (5.8)) appear. Of course, a sum over all bracket arrangements would be a natural choice. But in fact, to find out whether this guess is correct, one needs a better understanding of the underlying A_∞-structure (cf. [14, 9]).

Since our results are consequences of restricting off-shell correlators to the known structure of conformal field theory it would be interesting to compare our results with other off-shell approaches, e.g. background independent open string field theory [16]–[19].[5]

[4] The total number of terms is always 14, which is the number of bracket arrangements for five functions.

[5] BSFT requires a fixed conformal background in the bulk, but allows for arbitrary boundary interactions. Our approach does not make reference to such a background and is closer to the spirit of the sigma model approach to string theory [22].

Clearly we expect that the tachyonic on-shell condition obtained in this paper is equivalent to the consistency condition for a string propagating in nontrivial background fields, i.e. the Weyl invariance condition of the underlying 2-d non-linear sigma model. Furthermore, the relation to previous work on the appearance of a non-commutative tachyon action, mostly considered in the limit of a strong magnetic field [20, 21], needs clarification. The exact non-commutative tachyon potential should be given by the well known expression obtained from BSFT but with ordinary products replaced with the generalized non-commutative product found in [8].

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